

A simple numerical algorithm for obtaining the longitudinal phase space density from the time projection of the current

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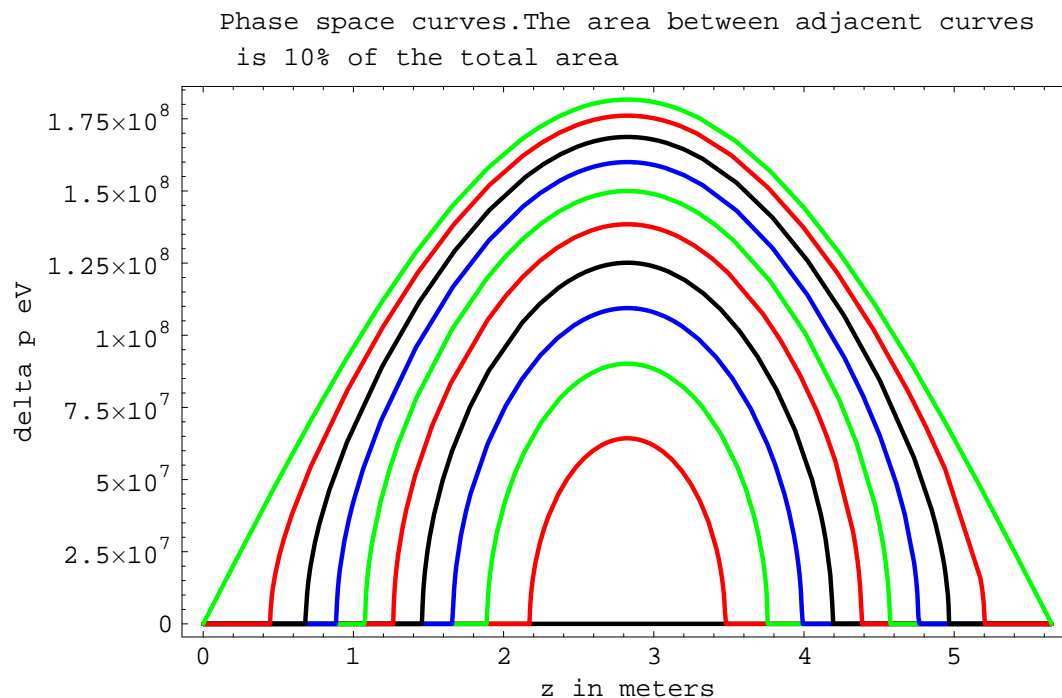
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Abstract

The purpose of this note is to outline a simple numerical algorithm to obtain the phase space density from the observed time dependence of the proton bunch current. It applies only to equilibrium distributions. It is implemented using Mathematica, but can be easily transformed to other platforms.

Approach

Consider the longitudinal phase space plot shown in the figure below.



We know from Liouville's Theorem that the density along any of the trajectories shown will be constant. Thus if we wish the density at $\rho(z, p)$, it can be obtained from $\rho(z_0, p_0)$ where z_0 is at the synchronous phase, $\lambda/2$, and p_0 is on the same trajectory as (z, p) . This is just a reflection of the fact that for an equilibrium distribution, there is only one variable on which the density

depends. The natural variable to pick is the “action” or the area of the associated phase space “ellipse”.

The connection between the two points is easy to calculate because the Hamaltonian is constant:

$$(1.1) \quad p^2 + \frac{2v^2 E_s e V_{rf}}{\eta h c^2} \cos(2\pi z / \lambda) = p_0^2 + \frac{2v^2 E_s e V_{rf}}{\eta h c^2} \cos(2\pi z_0 / \lambda)$$

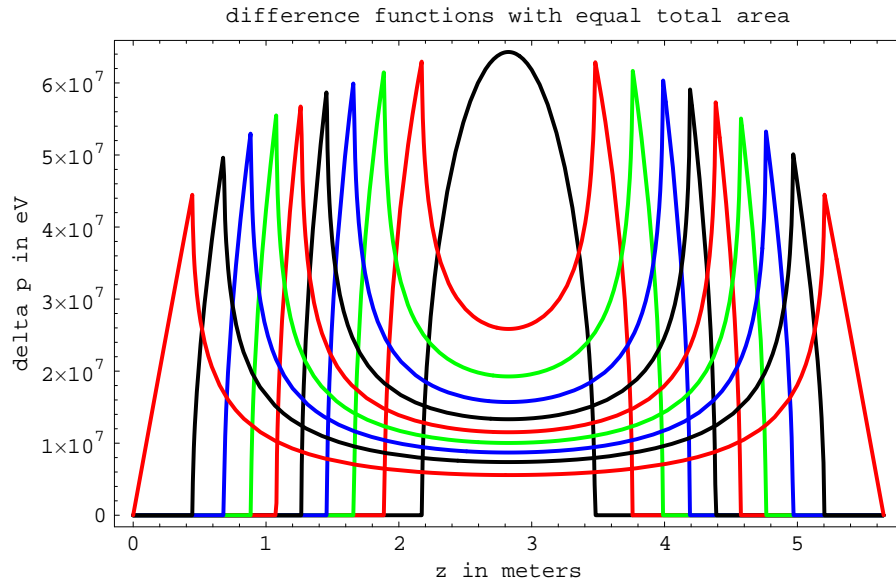
where p is the momentum deviation, λ is the wavelength, E_s is the energy, η is the slip factor, h is the harmonic number and V_{rf} is the cavity voltage. Applying this to the case where z_0 is on the center line:

$$(1.2) \quad p_0^2 = p^2 + \frac{2v^2 E_s e V_{rf}}{\eta h c^2} (\cos(2\pi z / \lambda) + 1)$$

Thus the problem of finding the density function becomes one of finding the density along the p axis at the synchronous phase.

The expansion functions

The figure above was drawn with the area difference between curves 10% of the total area or in steps of 10% of the action. Approximately the density will be constant in these bands. This leads us to propose using these difference functions as a set of functions in which to expand the proton current. Since the density is constant, the total number of protons in a slice of z is just proportional to the height of the individual curves. The difference functions shown below correspond to the first figure.



The figure shows the case for ten curves, but in fact we will use of the order of fifty. The curves have equal areas and so we need a recipe to calculate the curves, ie we need to associate each curve with its action which is the average value of the action for the two bounding phase space curves. We need a formula that connects the action S with its corresponding phase curve. But the phase space curve is completely determined by specifying the value of p_0 on the central axis. So the relation we seek is given by using eq.1.2 above:

$$(1.3) \quad S = \int \sqrt{p_0^2 - \frac{2v^2 E_s e V_{rf}}{\eta h c^2} (\cos(2\pi z / \lambda) + 1)} dz$$

This is an elliptic integral that is easily done and gives $S(p_0)$. In a practical sense it easiest to integrate over z from zero to λ and take the real part after the integral is done. However, the relation we have has to be inverted so that we have $p_0(S)$ so that we can generate the equally spaced curves in action, S . This is done easily by making a numerical table of S vs p_0 , and from this table generate an interpolation function that gives $p_0(S)$. Once this function is known, the curves are easily generated.

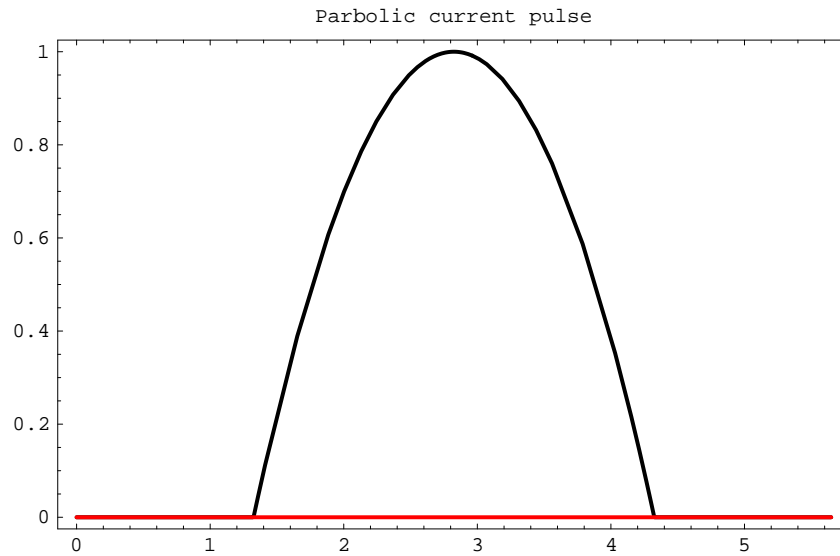
If we designate the difference functions by $F_i(z)$, then we have

$$(1.4) \quad I_p(z) = \sum_i C_i F_i(z)$$

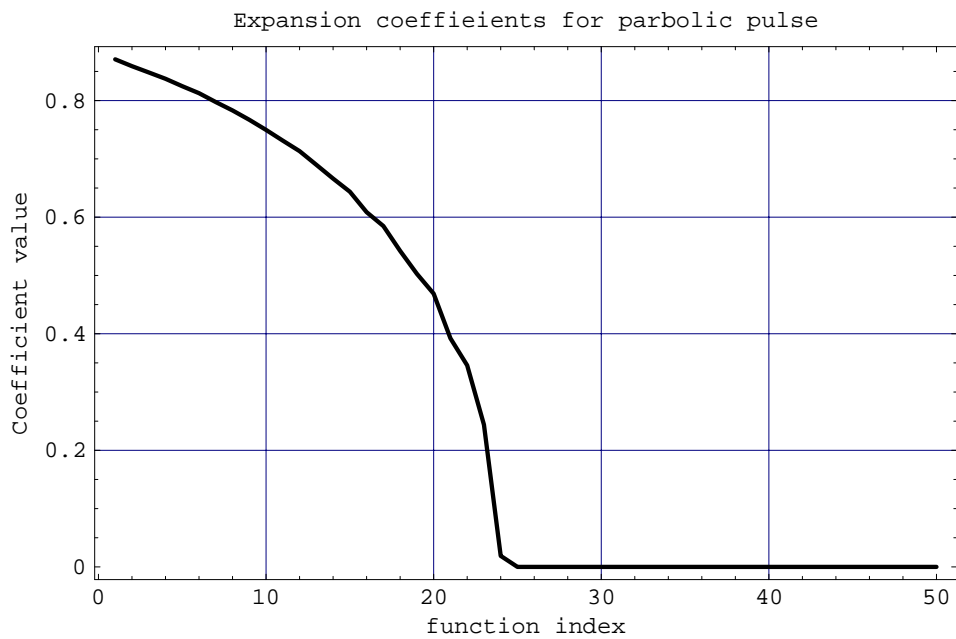
The C_i then give the density as a function of the index i which is directly related to the action and the density along the central axis. The constants are made by breaking up the proton current pulse into a number of discrete points and making a least squares fit using eq. 1.4.

A toy example

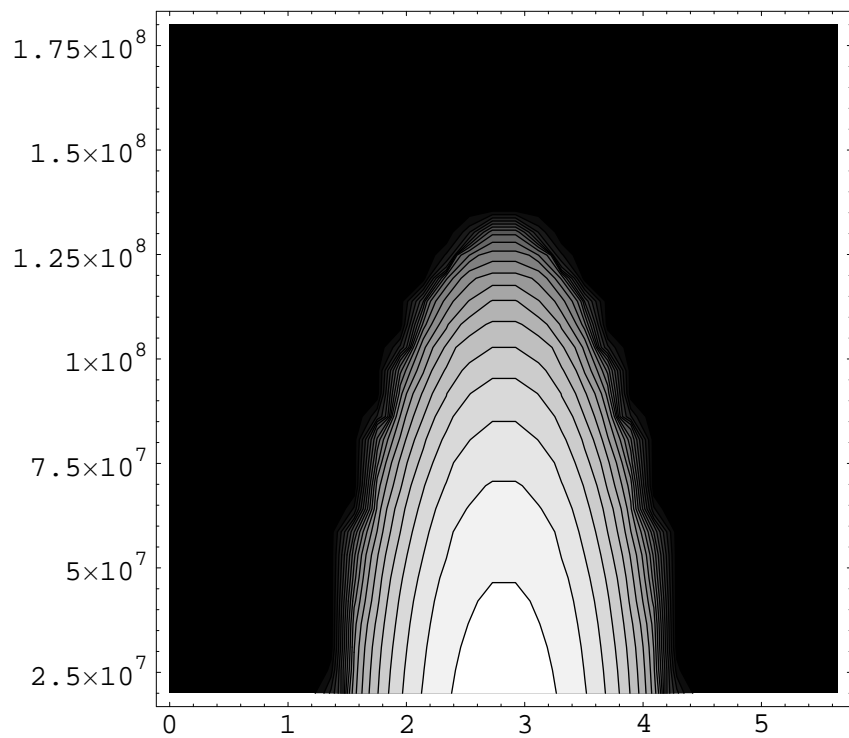
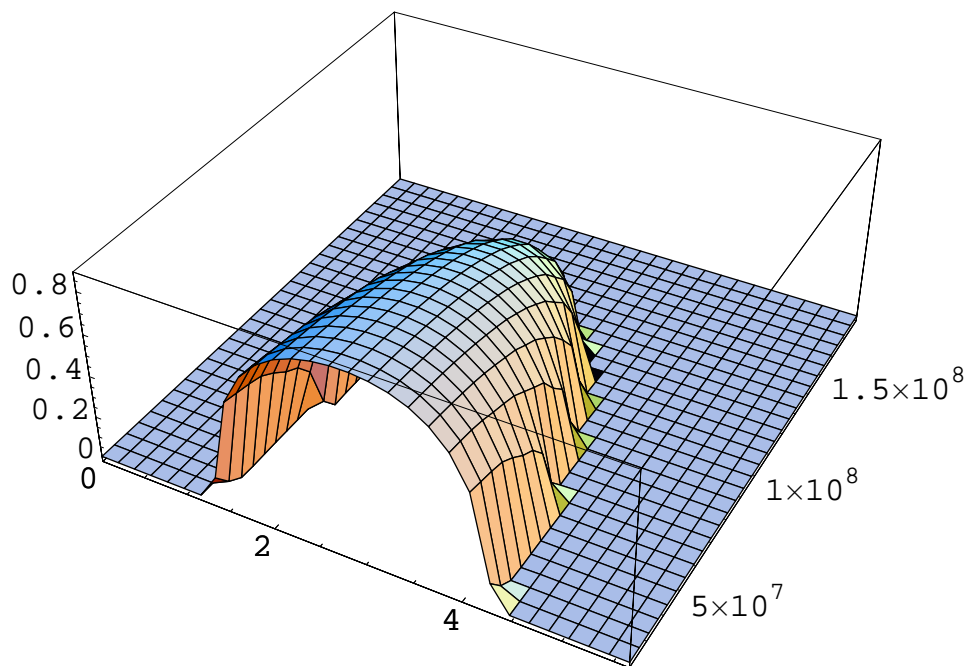
As an example, we take a parabolic pulse that has a base width of about half the bucket width. This is shown in the figure below. We use a set of expansion functions that are 2% of the total phase space area, and we have broken up the first half of the pulse into 50 points. Note that it is necessary to have more points than fitting functions. We will come back to this point later.



The expansion yields the following coefficients:

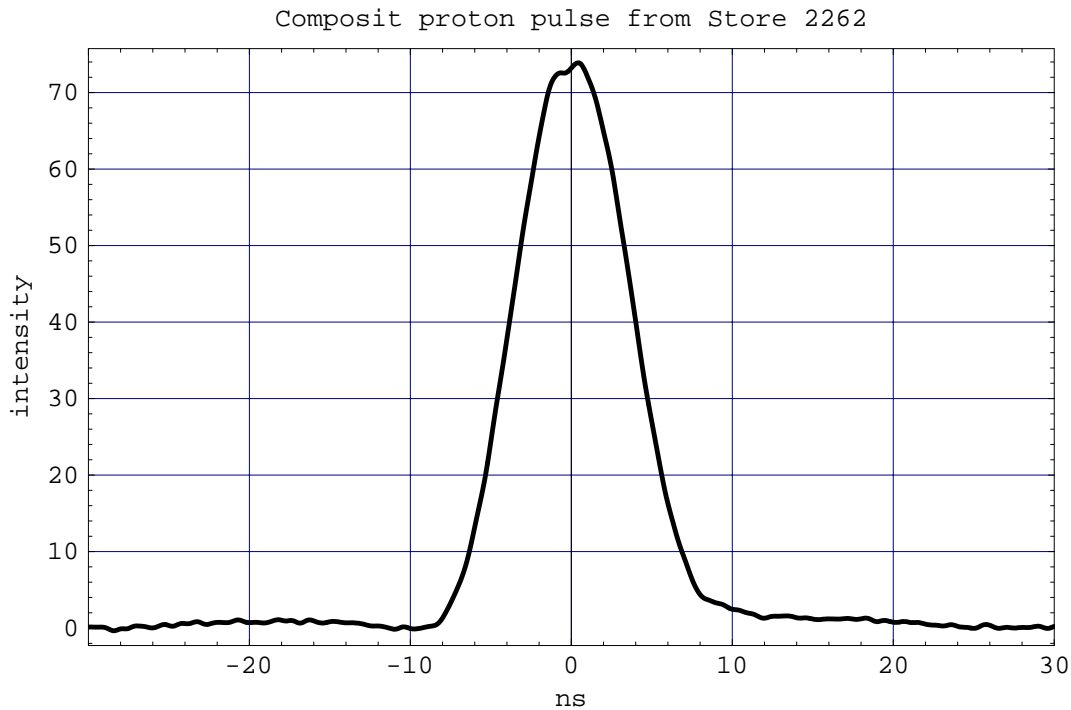


None of the higher order functions are necessary because there is no current in the region from $z=0$ to about 1.3 meters. The following two figures show the density as a 3D plot and as a contour plot. A plot of the input parabola and the output fitted function agree to about 1% in this case. In fact plotting the two curves on top of each other yields only a single line.

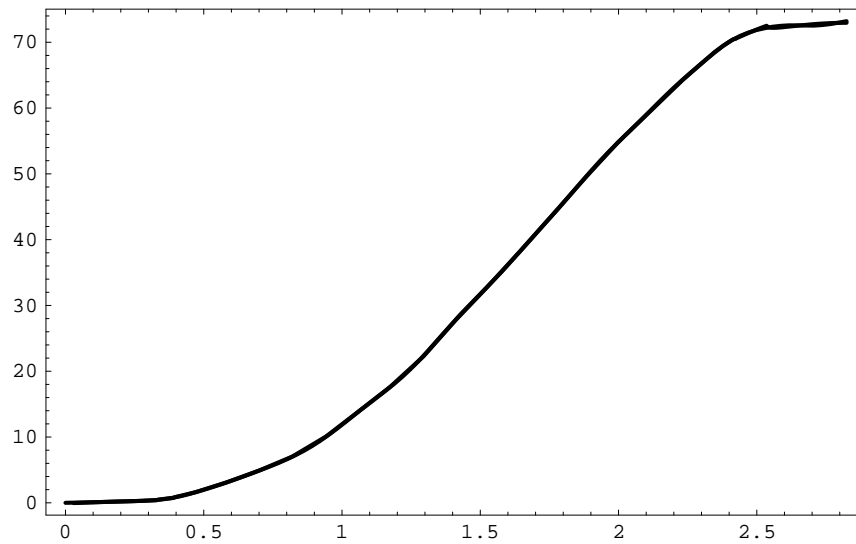


Composit Tevatron proton pulse at 150 GeV/c

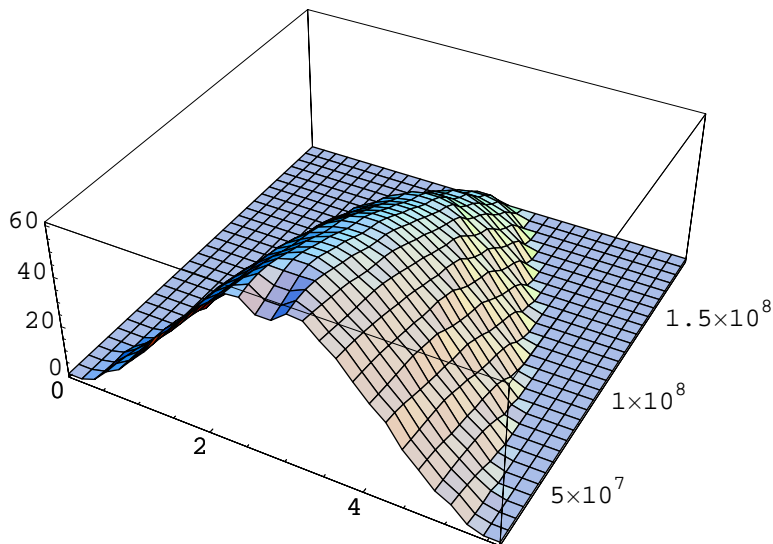
As a second example, we use data from the Tevatron at injection. The pulses are wide, nearly filling the available bucket. The wall current monitor digitizes at 0.5 ns intervals which yields about 37 points per bucket. It is clear that none of the 36 proton pulses represent a phase space that is in complete equilibrium. There are local blobs that move around with the synchrotron frequency. An ensemble is generated by superimposing all 36 bunches on top of each other. The technique used was to fit each of the 36 bunches with a local Gaussian to get the centroid. Next, a local interpolation (3rd order) function was generated for each of the 36 pulses. Using the time from the Gaussian fit, these bunches were superimposed to give the ensemble. It is shown below.



There is a small satellite at -19 ns, in the adjacent bucket, and a combination of a dispersive tail due to the cable from the tunnel to F0 and perhaps a small following satellite. The dispersion in the cable can be corrected, but for this study, I have taken just the leading half of the pulse. {It is clear that there is not complete statistical equilibrium because one sees a small bump on the right side of the peak}. Since this pulse is generated by adding together 36 interpolation functions, it is continuous, and can be used to generate 50 data points in the region from -9.4 ns to 0.0. These points are then fit with our 50 F_i . The figure below shows the original half pulse and the fitted function superimposed. The horizontal scale has been changed to meters from the ns scale in the above figure. The fit is quite good. The 3D plot is also shown.

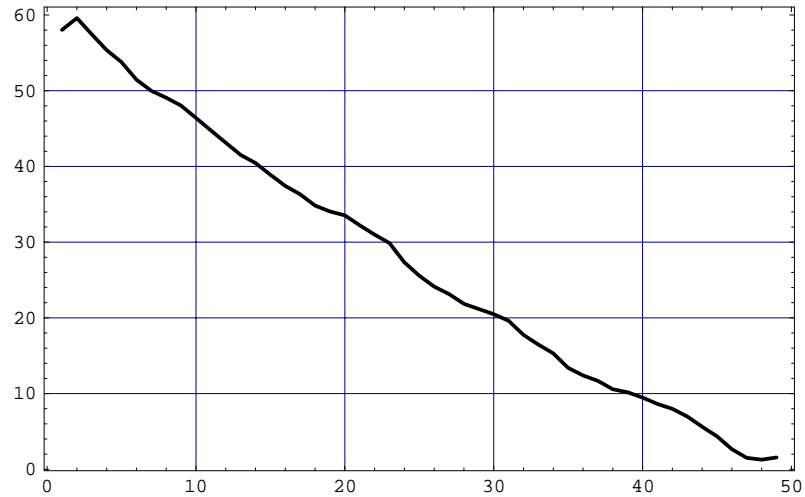


The original current (black) and the fit {blue) are shown in the figure above. The figure below is the reconstructed phase space.

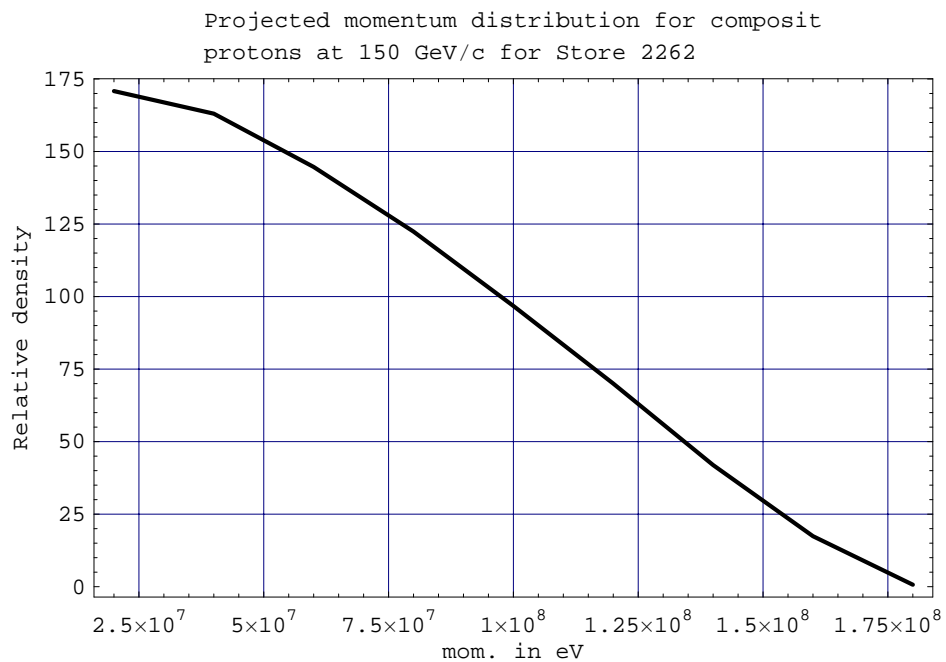


Notice the small dimple at the peak. This comes from injection errors. When a pulse is injected off center, it filaments into a ring leaving a hole in the center. The figure above indicates that on average there was an injection error for the ensemble. This is consistent with the current pulse which has a small dimple just after the center.

The coefficients are:



Given the coefficients, one can project the distribution along the z axis to get the momentum distribution.



This is less like a Gaussian than is the current pulse, and has a sigma that is smaller than would be predicted using a linear relation between z and p. Note that this reflects in the transverse

emittance calculation where the longitudinal size is subtracted in quadrature from the measured total sigma.

Conclusion

After the expansion functions and the interpolation functions are set up, the method is very fast as it only requires one least squares fit to obtain the answer. It gives limited information about the momentum distribution from the pulse shape. Good information for the input is necessary which the wall monitor can supply. The technique will be to take 20 to 30 sweeps and superimpose these to give good information on each of the 36 bunches. The present scope gives a t_0 for the sweep that furnishes a precise way to superimpose multiple pulses without a lot of digital processing. (The use of a Gaussian fit as described above is not necessary when t_0 is available. It was not available for the measurements described here.)

The use of numerical methods and interpolation functions greatly simplifies the problem and makes its solution transparent. Leo Michelotti [2] and Cheung-Yang Tan [1] have both written notes on this subject which are more analytical and make useful reading. The first mentioned note has a rather complete list of references and some discussion of the historical background.

References

1. C.-Y Tan A method for calculating longitudinal phase space distribution when given the time profile of the bunch. Technical Memo FERMILAB-TM-2155 July 2001.
2. Leo Michelotti. An integral for longitudinal phase space tomography on equilibrium distributions. FERMILAB-FN-0726 Oct. 2002.